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Dedicated to Professor Masafumi Yoshino for his sixtieth birthday

ON THE MULTISUMMABILITY OF WKB SOLUTIONS OF CERTAIN SINGULARLY PERTURBED LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Using two concrete examples, we discuss the multisummability of WKB solutions of singularly perturbed linear ordinary differential equations. Integral representations of solutions and a criterion for the multisummability based on the Cauchy-Heine transform play an important role in the proof.

Keywords: exact WKB analysis, WKB solution, multisummability.

Mathematics Subject Classification: 34M60, 34E20, 34M30, 40G10.

1. INTRODUCTION

In this paper we consider a singularly perturbed linear ordinary differential equation of the following form:

$$\left(\eta^{-m}\frac{d^m}{dz^m} + q_1(z,\eta^{-1})\eta^{-(m-1)}\frac{d^{m-1}}{dz^{m-1}} + \ldots + q_m(z,\eta^{-1})\right)\psi(z,\eta) = 0. \tag{1.1}$$

Here η is a large parameter and $q_j(z,\eta^{-1})$ $(1 \le j \le m)$ is a polynomial of z and η^{-1}, that is,

$$q_j(z,\eta^{-1}) = q_{j,0}(z) + \eta^{-1}q_{j,1}(z) + \eta^{-2}q_{j,2}(z) + \ldots \quad \text{(finite sum)}, \tag{1.2}$$

where $q_{j,k}(z)$ $(k = 0,1,\ldots)$ are polynomials of z. Equation (1.1) admits a formal solution of the form

$$\widehat{\psi}(z,\eta) = \exp\left(\eta\int^z \zeta(z)\,dz\right)\sum_{n=0}^{\infty}\psi_n(z)\eta^{-(n+1/2)}, \tag{1.3}$$

where $\zeta(z)$ is a root of the characteristic equation of (1.1):

$$\zeta^m + q_{1,0}(z)\zeta^{m-1} + \ldots + q_{m,0}(z) = 0. \tag{1.4}$$

A formal solution of this form is often called a WKB solution of (1.1). The purpose of this paper is to discuss the multisummability of a WKB solution of (1.1).

The most typical equation of the form (1.1) is the one-dimensional Schrödinger equation

$$\left(\eta^{-2}\frac{d^2}{dz^2} - Q(z)\right)\psi(z,\eta) = 0. \tag{1.5}$$

In this case a WKB solution can be expressed as

$$\widehat{\psi}_{\pm}(z,\eta) = \exp\left(\pm\eta\int^z \sqrt{Q_0(z)}\,dz\right)\sum_{n=0}^{\infty}\psi_{\pm,n}(z)\eta^{-(n+1/2)} \tag{1.6}$$

and, as is well-known, a WKB solution (1.6) is divergent in almost all cases. In the exact WKB analysis initiated by Voros ([10]) the Borel summation technique is employed to endow WKB solutions with an analytic meaning and the global behavior of solutions of (1.5) (e.g., the monodromy group, Stokes multipliers around irregular singular points, etc.) is successfully analyzed in an explicit manner by using Borel resummed WKB solutions. (See, for example, [3, 5].) For the Borel summability of WKB solutions of (1.5) we refer the readers to [2, 4, 6] and references cited there.

However, if we deal with a more general equation of the form (1.1) (for example, if some perturbative terms (with respect to η^{-1}) are added to the potential $Q(z)$ in (1.5) like $Q(z,\eta^{-1}) = Q_0(z) + \eta^{-1}Q_1(z) + \ldots$), then it becomes necessary to consider the so-called multisummability to give an analytic meaning to WKB solutions in general. As a matter of fact, R. Schäfke ([7]) showed that the following first-order inhomogeneous ordinary differential equation

$$\left(\epsilon\frac{d}{dz} - \left(z - \epsilon z^2\right)\right)\psi(z,\epsilon) = \epsilon^2 \tag{1.7}$$

with a small parameter ϵ has a formal solution which is $(3,1)$-multisummable. Furthermore, inspired by this result, Suzuki considered an example of the perturbed Schrödinger equation of the form

$$\left(\eta^{-2}\frac{d^2}{dz^2} - \left(z - \eta^{-2}z^2\right)\right)\psi(z,\eta) = 0 \tag{1.8}$$

in his master thesis ([8]) and showed that a (suitably normalized) WKB solution of (1.8) is $(4,1)$-multisummable.

In this paper, as a generalization of their results, we discuss the multisummability of WKB solutions of a third-order homogeneous linear ordinary differential equation of the form (1.1). To be more specific, we consider

$$\left(\eta^{-3}\frac{d^3}{dz^3} + (z\eta^{-3})\eta^{-2}\frac{d^2}{dz^2} + (3+2z\eta^{-1})\eta^{-1}\frac{d}{dz} + 2i(z+1)\right)\psi(z,\eta) = 0 \tag{1.9}$$

as an example and show that (suitably normalized) WKB solutions $\widehat{\psi}(z,\eta)$ of (1.9) is $(8,5,1)$-multisummable (with respect to η).

In the paper, making use of an integral representation of solutions, we provide a complete proof of the multisummability for Equation (1.9) as well as that for Equation (1.8). The proof of the multisummability for (1.8) given below is slightly different from that of Suzuki ([8]). It is modified so that it becomes applicable to more general equations such as Equation (1.9). Although we here discuss only particular examples (1.8) and (1.9) to avoid complicated notations and to make the discussion more concise and definite, the reasoning employed in this paper can be easily generalized to more general equations of the form (1.1) as far as it has an integral representation of solutions. Thus we conclude that it is necessary to introduce the multisummability with several different indices to discuss the summability of WKB solutions of a singularly perturbed linear ordinary differential equation of the form (1.1) in general.

The paper is organized as follows: First we describe our main results in a specific manner in Section 2. Then, before proving the main results, we briefly review the definition of the multisummability in Section 3. Sections 4 and 5 are devoted to the proofs of the main results. In Appendices A and B we present several figures of steepest descent paths relevant to the proofs of the main results.

The main results of this paper were already announced in [9].

2. MAIN THEOREMS

Let us now state our main theorems in a more specific manner.

First, let us consider the second order equation discussed by Suzuki in [8]:

$$\left(\frac{d^2}{dz^2}-\eta^2\big(z-\eta^{-2}z^2\big)\right)\psi(z,\eta)=0, \tag{2.1}$$

which is a perturbation of the Airy equation. One characteristic feature of Equation (2.1) is that by the scaling

$$z=\eta^2 x \tag{2.2}$$

(2.1) is transformed to the Weber equation with a new large parameter $\zeta=\eta^4$:

$$\left(\frac{d^2}{dx^2}-(\eta^4)^2(x-x^2)\right)\psi=0. \tag{2.3}$$

To discuss the (multi)summability of WKB solutions of (2.1), we make full use of the integral representation of solutions for (2.1), which can be obtained in the following way: a change of unknown functions $\psi=\exp(-iz^2/2)\varphi$ transforms (2.1) to

$$\left(\frac{d^2}{dz^2}-2iz\frac{d}{dz}-\eta^2(z+\eta^{-2}i)\right)\varphi=0. \tag{2.4}$$

Since Equation (2.4) is of Laplace type, its integral representation of solutions can be easily constructed via Laplace transform, that is, letting $\varphi = \int \exp(-\eta z t)\hat{\varphi}(t)dt$, we find that $\hat{\varphi}(t)$ satisfies the following differential equation of first order:

$$(-\eta + 2it)\frac{d\hat{\varphi}}{dt} + (\eta^2 t^2 + i)\hat{\varphi} = 0. \tag{2.5}$$

Then, using an explicit form

$$\hat{\varphi} = \exp\left(\eta \int^t \frac{u^2 + i\eta^{-2}}{1 - 2iu\eta^{-1}} du\right) \tag{2.6}$$

of solutions of (2.5), we obtain an integral representation of solutions for the original equation (2.1):

$$\psi(z,\eta) = \int \exp\left(-\eta g(t; z, \eta^{-1})\right) dt, \tag{2.7}$$

where the phase function $g(t; z, \eta^{-1})$ is given by

$$g(t; z, \eta^{-1}) = zt - \int^t \frac{u^2 + i\eta^{-2}}{1 - 2iu\eta^{-1}} du + \frac{i}{2} z^2 \eta^{-1}. \tag{2.8}$$

Note that, by a change of variables $t = i\eta(s - 1/2)$, (2.8) can be written also as (a constant multiple of)

$$\psi = \int \exp\left(-\eta^4 \tilde{g}(s; z, \eta^{-1})\right) s^{-1/2}\, ds, \tag{2.9}$$

where

$$\tilde{g}(s; z, \eta^{-1}) = \frac{i}{8}\left(2s^2 - 4s(1-2x) + \log s + (1-2x)^2\right) \tag{2.10}$$

with $x = \eta^{-2} z$ (cf. (2.2)). The formula (2.10) is a well-known integral representation of solutions for the Weber equation (2.3).

Let $t = t_\pm$ be a saddle point of $g(t; z, \eta^{-1})$, that is, $t = t_\pm$ is a zero of

$$\frac{\partial g}{\partial t} = z - \frac{t^2 + i\eta^{-2}}{1 - 2it\eta^{-1}} = 0, \tag{2.11}$$

more explicitly,

$$t_\pm = -i\eta^{-1} z \mp \sqrt{z - \eta^{-2}(z^2 + i)}. \tag{2.12}$$

We also denote the top order part (with respect to η^{-1}) of g and $t_\pm$ by g_0 and $t_{\pm,0}$, respectively. Then,

$$g_0 = g_0(t, z) = zt - \frac{t^3}{3}, \quad t_{\pm,0} = \mp\sqrt{z} \quad \text{and} \quad \frac{\partial g_0}{\partial t}(t_{\pm,0}, z) = 0 \tag{2.13}$$

hold. Let $\Gamma_\pm$ be a steepest descent path of $\Re(-\eta g)$ passing through the saddle point $t_\pm$ and let $\psi_\pm(z,\eta)$ denote a solution of (2.1) defined by

$$\psi_\pm(z,\eta) = \int_{\Gamma_\pm} \exp\left(-\eta g(t; z, \eta^{-1})\right) dt = \exp\left(-\eta g_0(t_{\pm,0}, z)\right) \psi_\pm^{(0)}(z,\eta), \tag{2.14}$$

where

$$\psi_{\pm}^{(0)}(z,\eta) = \exp\left(-\eta(g(t_{\pm};z,\eta^{-1}) - g_0(t_{\pm,0},z))\right) \times \int_{\Gamma_{\pm}} \exp\left(-\eta(g(t;z,\eta^{-1}) - g(t_{\pm};z,\eta^{-1}))\right)\,dt. \tag{2.15}$$

Note that the exponential term of $\psi_{\pm}(z,\eta)$ satisfies

$$-\frac{d}{dz}\big(g_0(t_{\pm,0},z)\big) = -\frac{\partial g_0}{\partial t}(t_{\pm,0},z)\frac{dt_{\pm,0}}{dz} - \frac{\partial g_0}{\partial z}(t_{\pm,0},z) = -t_{\pm,0} = \pm\sqrt{z}. \tag{2.16}$$

We now consider the asymptotic expansion of the integral

$$\int_{\Gamma_{\pm}} \exp\left(-\eta(g(t;z,\eta^{-1}) - g(t_{\pm};z,\eta^{-1}))\right)\,dt \tag{2.17}$$

with respect to η (for fixed z). Since the contribution to the asymptotic expansion only comes from an arbitrarily small neighborhood of the saddle point $t = t_{\pm}$ and $g(t;z,\eta^{-1})$ is analytic (in η^{-1}) there, it suffices to discuss the integral of the form

$$\sum_{k=0}^{\infty}\eta^{-k} \int_{\text{along } \Gamma_{\pm},\ |t-t_{\pm}|\,:\,\text{small}} \exp\left(-\eta(g_0(t,z) - g_0(t_{\pm,0},z))\right) A_k(t,z)\,dt \tag{2.18}$$

where $A_k(t,z)$ is an analytic function of (t,z). Then, by applying the saddle point method to each coefficient of η^{-k} of (2.18) (or, by introducing a new variable $\theta = g_0(t,z) - g_0(t_{\pm,0},z)$ and applying Watson's lemma (cf., e.g., [1, §2.1, Theorem 1])), we find that $\psi_{\pm}^{(0)}(z,\eta)$ has an asymptotic expansion of the following form when $\eta \to \infty$:

$$\psi_{\pm}^{(0)}(z,\eta) \cong \widehat{\psi}_{\pm}^{(0)}(z,\eta) \overset{\text{def}}{=} \sum_{n=0}^{\infty}\psi_{\pm,n}(z)\eta^{-(n+1/2)}. \tag{2.19}$$

Furthermore, (2.19) holds in the sense of Gevrey order 1 (see in Section 3 below for the precise meaning of Gevrey asymptotics). Hence, in view of (2.16), we have

$$\psi_{\pm}(z,\eta) \cong \widehat{\psi}_{\pm}(z,\eta) \overset{\text{def}}{=} \exp\left(\pm\eta\int^{z}\sqrt{z}\,dz\right)\sum_{n=0}^{\infty}\psi_{\pm,n}(z)\eta^{-(n+1/2)}. \tag{2.20}$$

Since $\psi_{\pm}(z,\eta)$ is a solution of (2.1), its asymptotic expansion $\widehat{\psi}_{\pm}(z,\eta)$ also satisfies (2.1) formally. Hence it coincides with a (suitably normalized) WKB solution of (2.1).

The main result of Suzuki's paper [8] is concerned with this WKB solution $\widehat{\psi}_{\pm}(z,\eta)$.

Theorem 2.1 ([8]). *The formal power series part $\widehat{\psi}_{\pm}^{(0)}(z,\eta)$ of the WKB solution $\widehat{\psi}_{\pm}(z,\eta)$ of (2.1) is $(4,1)$-multisummable with respect to η^{-1}. To be more precise, for each fixed z, $\widehat{\psi}_{\pm}^{(0)}(z,\eta)$ is $(4,1)$-multisummable with respect to η^{-1} except for a finite number of singular directions.*

As the second example, let us next consider the following third-order differential equation:

$$\left(\frac{d^3}{dz^3} + (z\eta^{-3})\eta\frac{d^2}{dz^2} + (3+2z\eta^{-1})\eta^2\frac{d}{dz} + 2i(z+1)\eta^3\right)\psi(z,\eta) = 0, \tag{2.21}$$

with the characteristic equation

$$\zeta^3 + 3\zeta + 2i(z+1) = 0. \tag{2.22}$$

In parallel to the case of the first example (2.1), (2.21) admits the following two different scalings. Firstly, by the scaling $z = \eta^3 x_1$ and $\zeta_1 = \eta^5$, (2.21) is transformed to

$$\left(\frac{d^3}{dx_1^3} + (x_1\zeta^{-1/5})\zeta_1\frac{d^2}{dx_1^2} + (3\zeta_1^{-2/5} + 2x_1)\zeta_1^2\frac{d}{dx_1} + 2i(x_1 + \zeta_1^{-3/5})\zeta_1^3\right)\psi = 0 \tag{2.23}$$

and, secondly, by the scaling $z = \eta^5 x_2$ and $\zeta_2 = \eta^8$, (2.21) is transformed to

$$\left(\frac{d^3}{dx_2^3} + x_2\zeta_2\frac{d^2}{dx_2^2} + (3\zeta_2^{-1/2} + 2x_2)\zeta_2^2\frac{d}{dx_2} + 2i(x_2\zeta_2^{-1/8} + \zeta^{-3/4})\zeta_2^3\right)\psi = 0. \tag{2.24}$$

Similarly to (2.1), as Equation (2.21) itself is of Laplace type, (2.21) also has the following integral representation of solutions:

$$\psi(z,\eta) = \int \exp\left(-\eta h(t;z,\eta^{-1})\right)\,dt, \tag{2.25}$$

with

$$h(t;z,\eta^{-1}) = zt - \int^t \frac{u^3 + (3-2\eta^{-4})u - 2i + 2\eta^{-2}}{\eta^{-3}u^2 - 2\eta^{-1}u + 2i}\,du. \tag{2.26}$$

Note that, by a change of variables $t = \eta^2 s$, (2.25) can be written also as

$$\psi = \int \exp\left(-\eta^8\tilde{h}(s;z,\eta^{-1})\right)\,ds, \tag{2.27}$$

where

$$\tilde{h}(s;z,\eta^{-1}) = x_2 s - \int^s \frac{v^3 + (3\eta^{-4} - 2\eta^{-8})v - 2i\eta^{-6} + 2\eta^{-8}}{v^2 - 2v + 2i\eta^{-1}}\,dv \tag{2.28}$$

with $x_2 = \eta^{-5}z$.

In the case of Equation (2.21) there exist three saddle points of $h(t;z,\eta^{-1})$, which are denoted by $t = t_j$ $(j = 1,2,3)$. Denoting the top order part of h and t_j by h_0 and $t_{j,0}$, respectively, we find that

$$\frac{\partial h_0}{\partial t}(t,z) = -\frac{i}{2}\left(-t^3 - 3t + 2i(z+1)\right) \quad \text{and} \quad \frac{\partial h_0}{\partial t}(t_{j,0},z) = 0 \tag{2.29}$$

hold. Let Γ_j $(j = 1, 2, 3)$ be a steepest descent path of $\Re(-\eta h)$ passing through the saddle point $t = t_j$ and let $\psi_j(z, \eta)$ be a solution of (2.21) defined by

$$\psi_j(z,\eta) = \int_{\Gamma_j} \exp\left(-\eta h(t; z, \eta^{-1})\right)\, dt = \exp\left(-\eta h_0(t_{j,0}, z)\right) \psi_{\pm}^{(0)}(z,\eta), \tag{2.30}$$

where

$$\begin{aligned}\psi_{\pm}^{(0)}(z,\eta) &= \exp\left(-\eta(h(t_j; z, \eta^{-1}) - h_0(t_{j,0}, z))\right) \\ &\quad \times \int_{\Gamma_j} \exp\left(-\eta(h(t; z, \eta^{-1}) - h(t_j; z, \eta^{-1}))\right)\, dt.\end{aligned} \tag{2.31}$$

In view of (2.22) and (2.29), we can readily confirm that the derivative (with respect to z)

$$-\frac{d}{dz}\left(h_0(t_{j,0}, z)\right) = -\frac{\partial h_0}{\partial t}(t_{j,0}, z)\frac{dt_{j,0}}{dz} - \frac{\partial h_0}{\partial z}(t_{\pm,0}, z) = -t_{j,0} \tag{2.32}$$

of the exponential term of $\psi_j(z, \eta)$ is a root of the characteristic equation (2.22). Then, by the same reasoning as above for the solution $\psi_{\pm}(z, \eta)$ of (2.1), we obtain a WKB solution $\widehat{\psi}_j(z, \eta)$ of (2.21) through the asymptotic expansion of $\psi_j(z, \eta)$:

$$\psi_j(z,\eta) \cong \widehat{\psi}_j(z,\eta) \stackrel{\text{def}}{=} \exp\left(\eta \int^z \zeta_j(z)\, dz\right) \sum_{n=0}^{\infty} \psi_{j,n}(z)\eta^{-(n+1/2)}, \tag{2.33}$$

where $\zeta_j(z)$ $(j = 1, 2, 3)$ is a root of the characteristic equation (2.22). Let $\widehat{\psi}_j^{(0)}(z, \eta)$ denote the formal power series part of $\widehat{\psi}_j(z, \eta)$:

$$\widehat{\psi}_j(z,\eta) = \exp\left(\eta \int^z \zeta_j(z)\, dz\right) \widehat{\psi}_j^{(0)}(z,\eta). \tag{2.34}$$

Our second main theorem is then the following:

Theorem 2.2. *The formal power series part $\widehat{\psi}_j^{(0)}(z, \eta)$ $(j = 1, 2, 3)$ of the WKB solution $\widehat{\psi}_j(z, \eta)$ of (2.21) is $(8, 5, 1)$-multisummable with respect to η^{-1}.*

Thus, to describe the multisummability of WKB solutions of (2.21), we need two other different indices 8 and 5 in addition to the index 1.

In what follows we give a proof of Theorems 2.1 and 2.2.

3. BRIEF REVIEW OF THE MULTISUMMABILITY

As a preparation for the proof of the main theorems, following [1], we review the definition and some fundamental properties for the multisummability in this section. We basically employ the same notation as in [1] except that we use a large parameter η here instead of a small parameter $\epsilon = \eta^{-1}$ as an asymptotic parameter.

First, let us recall the definition of the k-summability.

Definition 3.1 (k-summability)**.** Let $k > 0$ be a positive real number and $\widehat{f} = \sum_n f_n \eta^{-n}$ be a formal power series of η^{-1}. Then $\widehat{f}$ is said to be k-summable in the direction d if and only if $\mathcal{L}_k^d \widehat{\mathcal{B}}_k \widehat{f}$ is well-defined.

Here $\widehat{\mathcal{B}}_k \widehat{f}$ denotes the formal k-Borel transform of $\widehat{f}$:

$$\left(\widehat{\mathcal{B}}_k \widehat{f}\right)(y) = \sum_{n=0}^{\infty} \frac{f_n}{\Gamma(1+n/k)} y^n, \tag{3.1}$$

and $\mathcal{L}_k^d g$ denotes the k-Laplace transform of g in the direction d:

$$\left(\mathcal{L}_k^d g\right)(\eta) = \eta^k \int_0^{\infty e^{id}} \exp\left(-(y\eta)^k\right) g(y)\, d(y^k), \tag{3.2}$$

where the integration from 0 to ∞ is done along $\arg y = d$. Note that the Borel summability is nothing but the 1-summability under this terminology.

It is known (cf., e.g., [1, §3.1, Theorem 1]) that the k-summability of $\widehat{f}$ is equivalent to the existence of an analytic function $f(\eta)$ whose Gevrey asymptotic expansion of order k is given by $\widehat{f}$ in a sector

$$S = S(d, \alpha, \rho) = \{\eta \in \mathbb{C} \mid d - \alpha/2 < \arg \eta < d + \alpha/2,\ |\eta| > \rho^{-1}\}$$

with $\alpha > \pi/k$:

$$f(\eta) \cong_k \widehat{f} = \sum_{n=0}^{\infty} f_n \eta^{-n} \quad \text{as } \eta \to \infty \quad \text{in } S, \tag{3.3}$$

that is, for every closed subsector $\overline{S_1}$ of S and every non-negative integer N

$$|\eta|^N \left| f(\eta) - \sum_{n=0}^{N-1} f_n \eta^{-n} \right| \le C K^N \Gamma(1 + N/k) \tag{3.4}$$

holds in $\eta \in \overline{S_1}$ with positive constants $C, K > 0$ independent of N.

Next, we recall the definition of the multisummability.

Definition 3.2 (multisummability)**.** Let $k = (k_1, \ldots, k_q)$ be a q-tuple of positive real numbers $\{k_j\}$ $(1 \le j \le q)$ satisfying $k_1 > k_2 > \ldots > k_q > 0$ and $\widehat{f} = \sum_n f_n \eta^{-n}$ be a formal power series of η^{-1}. Then $\widehat{f}$ is said to be k-multisummable in the direction d if and only if the following functions $\{f_j\}$ $(0 \le j \le q)$ are successively well-defined:

$$\begin{array}{rcl} f_q & := & \widehat{\mathcal{B}}_{k_q} \widehat{f}, \\ f_{q-1} & := & \mathcal{A}^d_{k_{q-1}, k_q} f_{k_q}, \\ & \ldots & \\ f_1 & := & \mathcal{A}^d_{k_1, k_2} f_2, \\ f_0 & := & \mathcal{L}^d_{k_1} f_1. \end{array} \tag{3.5}$$

Here $\mathcal{A}^d_{\tilde{k},k} = \mathcal{B}_{\tilde{k}} \circ \mathcal{L}^d_k$ denotes the acceleration operator introduced by Ecalle, that is,

$$\left(\mathcal{A}^d_{\tilde{k},k} g\right)(\eta) = \eta^k \int_0^{\infty e^{id}} C_{\tilde{k}/k}\left((y\eta)^k\right) g(y)\, d(y^k), \tag{3.6}$$

where the integration is done along $\arg y = d$ from 0 to ∞ and the kernel function $C_\alpha(z)$ $(\alpha > 1)$ is given as follows:

$$C_\alpha(z) = \frac{1}{2\pi i} \int_\gamma u^{1/\alpha - 1} \exp\left(u - z u^{1/\alpha}\right) du, \tag{3.7}$$

where γ is a path going from $-\infty$ to $-\delta$ $(\delta > 0)$ along the negative real axis, encircling the origin anti-clockwise once, and returning to $-\infty$ again along the negative real axis. When $\widehat{f}$ is k-summable, the function f_0 defined by (3.5) is called *the k-sum of $\widehat{f}$*.

Remark 3.3. The multisummability is usually defined in the (admissible) multi-direction $d = (d_1, \ldots, d_q)$, that is, in the definition of f_j in (3.5), we use different directions d_j as $f_{j-1} = \mathcal{A}^{d_j}_{k_{j-1},k_j} f_{k_j}$ for $2 \le j \le q$ and $f_0 = \mathcal{L}^{d_1}_{k_1} f_1$. In this paper, however, we only consider the multisummability in a fixed single direction d for the sake of simplicity.

Roughly speaking, the multisummability deals with the situation where we need to consider the k_j-summability with several different indices k_j simultaneously, as is clearly shown by the following characterization of the multisummable series.

Proposition 3.4 ([1, §6.2 and §6.3])**.** *Suppose $k_q > 1/2$. Then a formal power series $\widehat{f}$ is $(k_1, \ldots, k_q)$-multisummable in the direction d if and only if $\widehat{f}$ can be decomposed into the sum of k_j-summable series $\widehat{f}_j$ in the direction d, that is,*

$$\widehat{f} = \sum_{j=1}^q \widehat{f}_j, \quad \text{where } \widehat{f}_j\text{: } k_j\text{-summable in } d. \tag{3.8}$$

The following criterion for the multisummability is also very useful in verifying the multisummability of a given formal power series.

Proposition 3.5 ([1, §6.7, Proposition 3])**.** *Let a formal power series $\widehat{f} = \sum_n f_n \eta^{-n}$ of η^{-1} be given. Let I_j $(1 \le j \le q)$ be closed intervals $I_j = [d - \pi/(2k_j), d + \pi/(2k_j)]$ so that $d \in I_1 \subset I_2 \subset \ldots \subset I_q$ holds, and assume $k_q > 1/2$. Suppose that there exist $\epsilon > 0$, $\rho > 0$ and φ_0 with $I_q \subset [\varphi_0, \varphi_0 + 2\pi]$ such that the following holds: For any θ with $\varphi_0 \le \theta \le \varphi_0 + 2\pi$ we may find $f(\eta;\theta)$ satisfying*

(i) *$f(\eta;\theta)$ is analytic in $S_\theta = S(\theta, \epsilon, \rho)$,*
(ii) *$f(\eta;\theta)$ is bounded as $\eta \to \infty$,*
(iii) *for every θ_1 and θ_2 with $|\theta_1 - \theta_2| < \epsilon$ (i.e., $S_{\theta_1} \cap S_{\theta_2} \ne \emptyset$) the following holds: If $\theta_1, \theta_2 \in I_1$, then*

$$f(\eta;\theta_1) = f(\eta;\theta_2). \tag{3.9}$$

If $\theta_1, \theta_2 \in I_j$ for some j, $2 \leq j \leq q$, then

$$f(\eta;\theta_1) - f(\eta;\theta_2) \cong_{k_{j-1}} 0 \quad \text{in } S_{\theta_1} \cap S_{\theta_2}. \tag{3.10}$$

If either θ_1 or θ_2 is not in I_q, then

$$f(\eta;\theta_1) - f(\eta;\theta_2) \cong_{k_q} 0 \quad \text{in } S_{\theta_1} \cap S_{\theta_2}. \tag{3.11}$$

(iv) $f(\eta;\varphi_0) = f(\eta e^{2\pi i};\varphi_0 + 2\pi)$ *in* S_{φ_0},
(v) $f(\eta;d) \cong_{k_q} \widehat{f}$ *in* S_d.

Then $\widehat{f}$ is $(k_1,\ldots,k_q)$-multisummable in the direction d.

Remark 3.6. Proposition 3.5 is proved by using the so-called Cauchy-Heine transform and Proposition 3.4. For the Cauchy-Heine transform see [1, Chap. 4]. Note also that Proposition 3.5 still holds if we replace conditions (3.10) and (3.11) by

$$f(\eta;\theta_1) - f(\eta;\theta_2) = O\Big(\exp\left(-c|\eta|^{k_{j-1}}\right)\Big) \quad \text{in } S_{\theta_1} \cap S_{\theta_2} \text{ for some } c > 0 \qquad (3.10)'$$

and

$$f(\eta;\theta_1) - f(\eta;\theta_2) = O\Big(\exp\left(-c|\eta|^{k_q}\right)\Big) \quad \text{in } S_{\theta_1} \cap S_{\theta_2} \text{ for some } c > 0, \qquad (3.11)'$$

respectively. In what follows, we use Proposition 3.5 in this modified form.

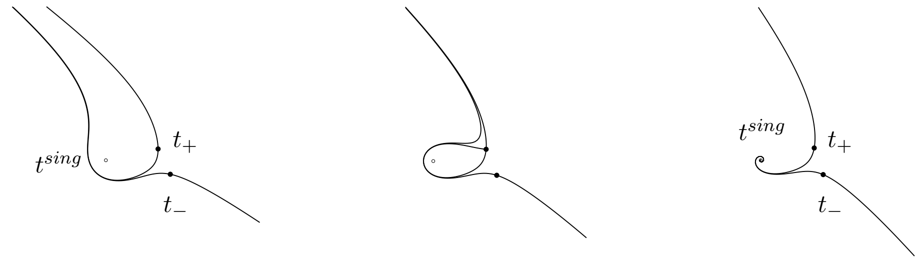
4. STRUCTURE OF STOKES PHENOMENA FOR WKB SOLUTIONS OF (2.1) AND (2.21)

One of the key steps in the proof of the main theorems is to investigate what kinds of Stokes phenomena occur with WKB solutions when $\arg\eta$ varies from 0 to 2π for fixed z. In the current situation, as there exist integral representations of solutions, this can be explicitly done by analyzing the change of the configuration of the steepest descent paths. In this section, examining the configuration of the steepest descent paths with the aid of a computer, we study the structure of Stokes phenomena for WKB solutions of (2.1) and (2.21).

4.1. CASE OF (2.1)

In the case of Equation (2.1) the structure of Stokes phenomena for WKB solutions was investigated in [8] in a detailed manner. We first review the results of [8] in this subsection.

For the sake of definiteness, we fix z as $z = 1 + i$ in what follows. Figure 1 shows the configuration of the steepest descent paths $\Gamma_\pm$ passing through the saddle points $t = t_\pm$ of the integral representation (2.7) near $\arg\eta = 0$. In Figure 1 we take $|\eta| = 10$ and a unique singular point $t = -i\eta/2$ is designated by t^{sing}. (In writing Figure 1, we use the integral representation (2.9) instead of (2.7), since $|t^{sing}|$ becomes too large in the original integral representation (2.7). As these two integrals are related by a simple change of integration variable $t = i\eta(s - 1/2)$, i.e., by a scaling and a translation, they are completely equivalent.)



$\arg\eta = -\pi/50$ $\arg\eta = 0$ $\arg\eta = \pi/50$

Fig. 1. Configuration of steepest descent paths of (2.9) near $\arg\eta = 0$

Figure 1 clearly visualizes that the configuration of the steepest descent paths $\Gamma_\pm$ for $\arg\eta < 0$ is different from that for $\arg\eta > 0$. For example, Γ_- for $\arg\eta < 0$ goes to ∞ after emanating from t_- and encircling the singular point t^{sing} in a clockwise manner, while Γ_- for $\arg\eta > 0$ ends at t^{sing}. This change of the configuration of Γ_- causes a Stokes phenomenon for $\widehat{\psi}_-(z,\eta)$ (or $\widehat{\psi}_-^{(0)}(z,\eta)$) to occur in the following way: Let $\Gamma_\pm$ and $\widetilde{\Gamma}_\pm$ (resp., $\psi_\pm(z,\eta)$ and $\widetilde{\psi}_\pm(z,\eta)$) denote the steepest descent paths passing through $t_\pm$ (resp., the corresponding solutions of (2.1) defined by (2.14)) for $\arg\eta < 0$ and $\arg\eta > 0$, respectively. Then it follows from Figure 1 that $\Gamma_- = \widetilde{\Gamma}_- + \widetilde{\Gamma}_\dagger$, where $\widetilde{\Gamma}_\dagger$ is a path emanating from $t = t^{sing}$ and going to ∞ through t_+, that is, a path homotopic to $\widetilde{\Gamma}_+$. Hence, we have

$$\psi_-(z,\eta) = \int\limits_{\Gamma_-} \exp\left(-\eta g(t;z,\eta^{-1})\right)\,dt = \widetilde{\psi}_-(z,\eta) + \int\limits_{\widetilde{\Gamma}_\dagger} \exp\left(-\eta g(t;z,\eta^{-1})\right)\,dt. \tag{4.1}$$

Note that, although $\widetilde{\Gamma}_\dagger$ is homotopic to $\widetilde{\Gamma}_+$, the second term of the right-hand side of (4.1) is not equal to $\widetilde{\psi}_+(z,\eta)$ since the branch of $g(t;z,\eta^{-1})$ on $\widetilde{\Gamma}_\dagger$ differs from that on $\widetilde{\Gamma}_+$ due to the singularity $t = t^{sing}$. Now for the phase function $g(t;z,\eta^{-1})$ of (2.7) we have

Lemma 4.1.
$$\operatorname*{Res}_{t=t^{sing}} \frac{\partial g}{\partial t}(t;z,\eta^{-1}) = \frac{i}{8}\eta^3(1-4i\eta^{-4}). \tag{4.2}$$

Hence, combining (4.1) and (4.2), we obtain

$$\psi_-(z,\eta) = \widetilde{\psi}_-(z,\eta) + \exp\left(-\frac{\pi}{4}\eta^4(1-4i\eta^{-4})\right)\widetilde{\psi}_+(z,\eta), \tag{4.3}$$

or, equivalently,

$$\begin{aligned} &\psi_-^{(0)}(z,\eta) - \widetilde{\psi}_-^{(0)}(z,\eta) \\ &\quad= \exp\left(-\frac{\pi}{4}\eta^4(1-4i\eta^{-4})\right)\exp\left(\eta(g_0(t_{-,0},z) - g_0(t_{+,0},z))\right)\widetilde{\psi}_+^{(0)}(z,\eta) \\ &\quad= O\big(\exp(-c|\eta|^4)\big) \end{aligned} \tag{4.4}$$

with some constant $c > 0$. Since $\psi_-^{(0)}(z,\eta)$ and $\widetilde{\psi}_-^{(0)}(z,\eta)$ are analytic realizations of $\widehat{\psi}_-^{(0)}(z,\eta)$ (i.e., the asymptotic expansion of both analytic solutions are given by $\widehat{\psi}_-^{(0)}(z,\eta)$) when $\arg\eta < 0$ and $\arg\eta > 0$, respectively, we thus conclude that a Stokes phenomenon of exponential order 4 occurs with $\widehat{\psi}_-^{(0)}(z,\eta)$ near $\arg\eta = 0$.

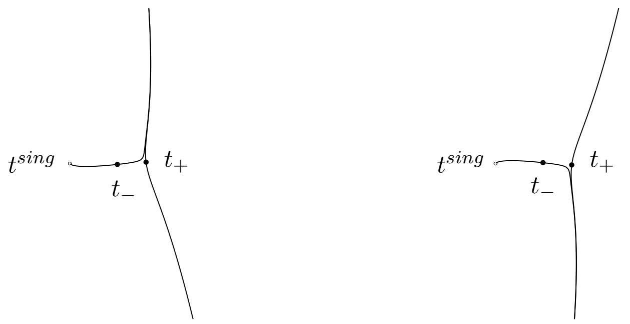
Similarly, it also follows from Figure 1 that $\Gamma_+ = \widetilde{\Gamma}_+ + \widetilde{\Gamma}_\dagger$ holds and hence we obtain

$$\psi_+^{(0)}(z,\eta) - \widetilde{\psi}_+^{(0)}(z,\eta) = \exp\left(-\frac{\pi}{4}\eta^4(1-4i\eta^{-4})\right)\widetilde{\psi}_+^{(0)}(z,\eta) = O\bigl(\exp(-c'|\eta|^4)\bigr) \tag{4.5}$$

with another constant $c' > 0$. Thus, a Stokes phenomenon of exponential order 4 also occurs with $\widehat{\psi}_+^{(0)}(z,\eta)$ near $\arg\eta = 0$.

Such Stokes phenomena of exponential order 4 occur near $\arg\eta = k\pi/4$ with $k = 0,1,\ldots,5$ for $\widehat{\psi}_-^{(0)}(z,\eta)$ and near $\arg\eta = k\pi/4$ with $k = -4,-3,\ldots,1$ for $\widehat{\psi}_+^{(0)}(z,\eta)$. This can be confirmed by tracing the configuration of steepest descent paths from $\arg\eta = 0$ to 2π. See Appendix A, where figures of steepest descent paths are given for several different values of $\arg\eta \in [0,2\pi)$.

On the other hand, a different kind of Stokes phenomenon occurs near $\arg\eta = 5\pi/8$. Figure 2 shows the configuration of steepest descent paths passing through $t = t_\pm$ near $\arg\eta = 5\pi/8$.



$\arg\eta = 12\pi/20$ $\arg\eta = 13\pi/20$

Fig. 2. Configuration of steepest descent paths of (2.9) near $\arg\eta = 5\pi/8$

Again let us denote by $\Gamma_\pm$ and $\widetilde{\Gamma}_\pm$ etc. the steepest descent paths passing through $t_\pm$ for $\arg\eta < 5\pi/8$ and $\arg\eta > 5\pi/8$, respectively. Then Figure 2 implies that $\Gamma_- = \widetilde{\Gamma}_- + \widetilde{\Gamma}_+$ and hence we have

$$\psi_-(z,\eta) = \widetilde{\psi}_-(z,\eta) + \widetilde{\psi}_+(z,\eta). \tag{4.6}$$

Note that there is no difference with the branch of $g(t;z,\eta^{-1})$ in this case. In other words, the singularity $t = t^{sing}$ is not relevant to the Stokes phenomenon near $\arg\eta = 5\pi/8$. Thus, we obtain

$$\psi_-(z,\eta) = \widetilde{\psi}_-(z,\eta) + \widetilde{\psi}_+(z,\eta), \tag{4.7}$$

or equivalently,

$$\begin{aligned}&\psi_-^{(0)}(z,\eta)-\widetilde{\psi}_-^{(0)}(z,\eta)\\&=\exp\left(\eta(g_0(t_{-,0},z)-g_0(t_{+,0},z)\right)\widetilde{\psi}_+^{(0)}(z,\eta)=O\big(\exp(-c''|\eta|)\big)\end{aligned}\tag{4.8}$$

with some constant $c''>0$, that is, near $\arg\eta=5\pi/8$ a Stokes phenomenon of exponential order 1 occurs with $\widehat{\psi}_-^{(0)}(z,\eta)$.

Summing up, we obtain the following proposition.

Proposition 4.2. *Let $z=1+i$ be fixed. Then, when $\arg\eta$ varies from 0 to 2π, the following two types of Stokes phenomena occur with $\widehat{\psi}_\pm^{(0)}(z,\eta)$.*

(type A)

$$\psi_*^{(0)}(z,\eta)-\widetilde{\psi}_*^{(0)}(z,\eta)=O(\exp(-c|\eta|)),\tag{4.9}$$

where $\psi_*^{(0)}(z,\eta)$ and $\widetilde{\psi}_*^{(0)}(z,\eta)$ denote the analytic realizations of $\widehat{\psi}_*^{(0)}(z,\eta)$ $(*=\pm)$ in neighboring two sectors, respectively, and c is a positive constant. This type of Stokes phenomena occurs near $\arg\eta=5\pi/8$ for $\widehat{\psi}_-^{(0)}(z,\eta)$, and near $\arg\eta=13\pi/8$ for $\widehat{\psi}_+^{(0)}(z,\eta)$.

(type B)

$$\psi_*^{(0)}(z,\eta)-\widetilde{\psi}_*^{(0)}(z,\eta)=O(\exp(-c|\eta|^4)).\tag{4.10}$$

This type of Stokes phenomena occurs at $\arg\eta=k\pi/4$ with $k=0,1,\ldots,5$ for $\widehat{\psi}_-^{(0)}(z,\eta)$, and at $\arg\eta=k\pi/4$ with $k=-4,-3,\ldots,1$ (modulo 2π) for $\widehat{\psi}_+^{(0)}(z,\eta)$.

Remark 4.3. As will become clear in the proof of Theorem 2.1 explained in Section 5.1, $(4,1)$-multisummability of WKB solutions of (2.1) follows from the occurrence of these two types Stokes phenomena with different exponential orders. In particular, the occurrence of Stokes phenomena of type B plays an important role and it is an immediate consequence of the fact that the residue of $\eta(\partial g/\partial t)$ at $t=t^{sing}$ is $O(\eta^4)$, i.e., of exponential order 4 with respect to η. This is also related to the existence of the scaling (2.2) that transforms (2.1) into (2.3).

Remark 4.4. In the proof of Proposition 4.2, to compute explicit values of $\arg\eta$ where Stokes phenomena occur numerically, we investigate the configuration of steepest descent paths by taking $|\eta|=10$. However, what we need to prove for Theorem 2.1 is the limiting value of $\arg\eta$ for $|\eta|\to\infty$. In this sense the argument in this section is an approximating one but, since the configuration of steepest descent paths depends continuously on η, we can deduce several important properties for the limiting value (for $|\eta|\to\infty$) of $\arg\eta$ where a Stokes phenomenon occurs from the above results.

For example, as is clear from the above argument, a Stokes phenomenon of type B occurs when saddle points are connected by a steepest descent path that encircles the singular point $t=t^{sing}$. In the limit $|\eta|\to\infty$ this is possible only when the top order part of $-2\pi i\eta\,\mathrm{Res}\,(\partial g/\partial t)$ with respect to η (i.e., the top order contribution to a contour integral around t^{sing}) becomes real, that is, when $\mathrm{Im}\ \eta^4=0$. This clearly explains why Stokes phenomena of type B occur only when $\arg\eta$ is an integral multiple of $\pi/4$ in Proposition 4.2.

Similarly, in the limit $|\eta| \to \infty$ it is naturally expected that a Stokes phenomenon of type A, which is of exponential order 1, is closely related to a Stokes phenomenon of the Airy equation $(d^2/dz^2 - \eta^2 z)\psi(z,\eta) = 0$. As is well known, Stokes phenomena of the Airy equation occur if and only if $\Im(\eta z^{3/2}) = 0$. Since $\arg z = \pi/4$ in the current situation of Proposition 4.2, this implies a Stokes phenomenon of the Airy equation occurs at $\arg \eta = 5\pi/8$ and $13\pi/8$. This is also consistent with the results of Proposition 4.2.

Remark 4.5. A Stokes phenomenon occurs only when a steepest descent path passing through a saddle point hits another saddle point. (Otherwise, a steepest descent path can be extended to infinity and $\widehat{\psi}_{\pm}^{(0)}(z,\eta)$ has an analytic realization there.) The most important consequence of Proposition 4.2 is that there are two types of Stokes phenomena for Equation (2.1), that is, a Stokes phenomenon of type A and that of type B: The former one (resp., the latter one) occurs when a steepest descent path hits another saddle point without encircling (resp., after encircling) the singular point $t = t^{sing}$. Since the distance between two saddle points of (2.7) is $O(\eta^0)$ while the distance between a saddle point and the singular point $t = t^{sing}$ is $O(\eta^1)$, in the limit $|\eta| \to \infty$ these two types of Stokes phenomena can be completely distinguished.

In the proof of Proposition 4.2, to visualize the configuration of steepest descent paths and to compute explicit values of $\arg \eta$ where Stokes phenomena occur, we have assumed $z = 1+i$, but this assumption is not essential: As the integral representation (2.7) exists for all values of z (z is just a parameter in the above investigation of (2.7)), the above two types of Stokes phenomena occur at several exceptional values of $\arg \eta$ for arbitrarily fixed z and except for such exceptional values of $\arg \eta$ steepest descent paths passing through saddle points can be extended to infinity and $\widehat{\psi}_{\pm}^{(0)}(z,\eta)$ has an analytic realization. (In general, the exact value of $\arg \eta$ where a Stokes phenomenon of type A occurs depends on z, while a Stokes phenomenon of type B occurs only when $\arg \eta$ is an integral multiple of $\pi/4$. See Remark 4.4 above.) The concrete numerical studies, which are possible for every $z \in \mathbb{C}$, are needed only to compute the explicit values of $\arg \eta$ where Stokes phenomena occur. Hence a proposition similar to Proposition 4.2 and consequently Theorem 2.1 can be considered to hold also for every $z \in \mathbb{C}$.

4.2. CASE OF (2.21)

In parallel to the preceding subsection, we investigate the structure of Stokes phenomena for WKB solutions of (2.21) by analyzing the change of the configuration of the steepest descent paths of the integral representation (2.25) in this subsection.

The integral representation (2.25) has two singular points at zeros of $\eta^{-3}t^2 - 2\eta^{-1}t + 2i = 0$, that is, at $t = \eta^2 \pm \sqrt{\eta^4 - 2i\eta^3}$. We will denote them by t_0^{sing} and t_1^{sing}:

$$\begin{cases} t_0^{sing} = \eta^2(1 + \sqrt{1 - 2i\eta^{-1}}) = 2\eta^2(1 + O(\eta^{-1})), \\ t_1^{sing} = \eta^2(1 - \sqrt{1 - 2i\eta^{-1}}) = i\eta(1 + O(\eta^{-1})). \end{cases} \tag{4.11}$$

The following lemma plays a crucially important role in the discussion hereafter.

Lemma 4.6.

$$\operatorname*{Res}_{t=t_0^{sing}} \frac{\partial h}{\partial t}(t; z, \eta^{-1}) = -4\eta^7\bigl(1 + O(\eta^{-1})\bigr). \tag{4.12}$$

$$\operatorname*{Res}_{t=t_1^{sing}} \frac{\partial h}{\partial t}(t; z, \eta^{-1}) = -\frac{i}{2}\eta^4\bigl(1 + O(\eta^{-1})\bigr). \tag{4.13}$$

We again fix z as $z = 1 + i$ for the sake of definiteness. Figures 3 and 4 show the configuration of the steepest descent paths Γ_j passing through the saddle points $t = t_j$ ($j = 1, 2, 3$) of the integral representation (2.25) near $\arg\eta = 0$. (In Figures 3 and 4 as well as in figures in Appendix B we take $|\eta| = 20$. As in the preceding subsection, instead of (2.25) we use the integral representation (2.27) since the singular points t_k^{sing} ($k = 0, 1$) becomes too large in (2.25). However, the use of (2.27) makes it difficult to distinguish the saddle points t_j ($j = 1, 2, 3$) and t_1^{sing}. Therefore, in presenting the configuration of steepest descent paths of (2.27), we also use two magnified figures, i.e., a magnified figure in the middle in Figures 3 and 4 so that t_1^{sing} may be distinguished from t_j, and a more magnified one in the right so that t_j may be distinguished from each other.)

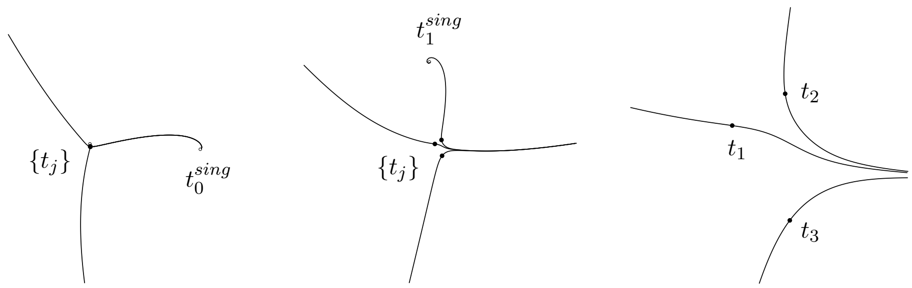


Fig. 3. Figure of the steepest descent paths of (2.27) at $\arg\eta = -3\pi/100$ and its magnification

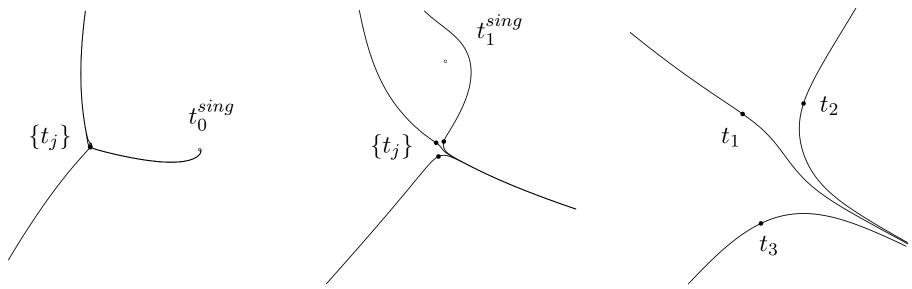


Fig. 4. Figure of the steepest descent paths of (2.27) at $\arg\eta = 3\pi/100$ and its magnification

As is clearly visualized in Figures 3 and 4, a change of the configuration of Γ_2 occurs near $\arg\eta = 0$, and consequently we have a Stokes phenomenon for $\widehat{\psi}_2(z,\eta)$ (or $\widehat{\psi}_2^{(0)}(z,\eta)$) as follows: Let Γ_j and $\widetilde{\Gamma}_j$ (resp., $\psi_j(z,\eta)$ and $\widetilde{\psi}_j(z,\eta)$) denote the steepest descent paths passing through t_j (resp., the corresponding solutions of (2.21) defined by (2.30)) for $\arg\eta < 0$ and $\arg\eta > 0$, respectively. In this case the change of the configuration of Γ_2 and the corresponding Stokes phenomenon for $\widehat{\psi}_2(z,\eta)$ are in a sense 'reversed' ones of those discussed in Section 4.1. Taking account of this character, we first observe that $\widetilde{\Gamma}_2 = \Gamma_2 + \Gamma_\dagger + \Gamma_\ddagger$, where $\Gamma_\dagger$ is a path emanating from $t = t_1^{sing}$ and going to ∞ through t_2 (i.e., homotopic to Γ_2) and $\Gamma_\ddagger$ is a path homotopic to Γ_1. Note again that the branch of $h(t;z,\eta^{-1})$ on $\Gamma_\dagger$ (resp., $\Gamma_\ddagger$) differs from that on Γ_2 (resp., Γ_1) due to the singularity $t = t_1^{sing}$. It then follows from Lemma 4.6 that

$$\widetilde{\psi}_2(z,\eta) = \psi_2(z,\eta) + O\big(\exp(-c\eta^5)\big)\psi_2(z,\eta) + O\big(\exp(-c\eta^5)\big)\psi_1(z,\eta) \tag{4.14}$$

for some constant $c > 0$. Since there occurs no Stokes phenomenon with $\psi_1(z,\eta)$ and $\psi_3(z,\eta)$, we finally obtain

$$\psi_2^{(0)}(z,\eta) - \widetilde{\psi}_2^{(0)}(z,\eta) = O\big(\exp(-c\eta^5)\big), \quad \psi_j^{(0)}(z,\eta) - \widetilde{\psi}_j^{(0)}(z,\eta) = 0 \quad (j = 1, 3), \tag{4.15}$$

that is, a Stokes phenomenon of exponential order 5 occurs with $\widehat{\psi}_2^{(0)}(z,\eta)$ near $\arg\eta = 0$.

Tracing the change of the configuration of the steepest descent paths of the integral representation (2.25) from $\arg\eta = 0$ to 2π, we thus conclude the following

Proposition 4.7. *Let $z = 1 + i$ be fixed. Then, when $\arg\eta$ varies from 0 to 2π, the following three types of Stokes phenomena occur with the WKB solutions $\widehat{\psi}_j^{(0)}(z,\eta)$ $(j = 1, 2, 3)$ of (2.21).*

(type A)

$$\psi_j^{(0)}(z,\eta) - \widetilde{\psi}_j^{(0)}(z,\eta) = O(\exp(-c|\eta|)), \tag{4.16}$$

where $\psi_j^{(0)}(z,\eta)$ and $\widetilde{\psi}_j^{(0)}(z,\eta)$ denote the analytic realizations of $\widehat{\psi}_j^{(0)}(z,\eta)$ in the neighboring two sectors, respectively, and c is a positive constant. This type of Stokes phenomena occurs near

$$\begin{cases} \arg\eta = 31\pi/100 \textit{ and } 164\pi/100 & \textit{for } \widehat{\psi}_1^{(0)}(z,\eta), \\ \arg\eta = 37\pi/100 \textit{ and } 62\pi/100 & \textit{for } \widehat{\psi}_2^{(0)}(z,\eta), \\ \arg\eta = 132\pi/100 \textit{ and } 138\pi/100 & \textit{for } \widehat{\psi}_3^{(0)}(z,\eta). \end{cases}$$

(type B)

$$\psi_j^{(0)}(z,\eta) - \widetilde{\psi}_j^{(0)}(z,\eta) = O(\exp(-c|\eta|^5)). \tag{4.17}$$

This type of Stokes phenomena occurs at $\arg\eta = k\pi/5$ with

$$\begin{cases} k = 1, 2, 3, 4 & \textit{for } \widehat{\psi}_1^{(0)}(z,\eta), \\ k = 0, 3, 7, 8, 9 & \textit{for } \widehat{\psi}_2^{(0)}(z,\eta), \\ k = 3, 4, 5, 6, 7, 8 & \textit{for } \widehat{\psi}_3^{(0)}(z,\eta). \end{cases}$$

(type C)

$$\psi_j^{(0)}(z,\eta) - \widetilde{\psi}_j^{(0)}(z,\eta) = O(\exp(-c|\eta|^8)). \tag{4.18}$$

This type of Stokes phenomena occurs at $\arg\eta = (2l+1)\pi/16$ *with*

$$\begin{cases} l = 0,1,3,4,6,9,10,11,12,15 & \textit{for } \widehat{\psi}_1^{(0)}(z,\eta), \\ l = 0,1,2,3,4,7,8,9,10,15 & \textit{for } \widehat{\psi}_2^{(0)}(z,\eta), \\ l = 0,3,4,5,6,13,14,15 & \textit{for } \widehat{\psi}_3^{(0)}(z,\eta). \end{cases}$$

For the change of the configuration of steepest descent paths see Appendix B, where figures of steepest descent paths of (2.25) or, equivalently, (2.27) together with their magnified ones are given for several different values of $\arg\eta \in [0,2\pi)$. Note that Stokes phenomena of type A are the ones to which both singular points $t = t_0^{sing}$ and $t = t_1^{sing}$ are irrelevant, while Stokes phenomena of type B (resp., type C) are the ones to which the singular point $t = t_1^{sing}$ (resp., $t = t_0^{sing}$) is relevant.

Remark 4.8. In view of Lemma 4.6, since a Stokes phenomenon of type B is the one to which the singular point $t = t_1^{sing}$ is relevant, we find that it occurs only when $\operatorname{Im}\eta^5 = 0$, that is, when the top order part (with respect to η) of $2\pi i$ multiple of the residue of $\eta(\partial h/\partial t)$ at $t = t_1^{sing}$ becomes real. This explains why Stokes phenomena of type B occur only when $\arg\eta$ is an integral multiple of $\pi/5$ in Proposition 4.7. Similarly, a Stokes phenomenon of type C occurs only when $\operatorname{Im} i\eta^8 = 0$, i.e., when $\arg\eta = (2l+1)\pi/16$ for some integer l.

Remark 4.9. In parallel to the case of Proposition 4.2, a proposition similar to Proposition 4.7 and consequently Theorem 2.2 can be considered to hold for every $z \in \mathbb{C}$ (except for exact values of $\arg\eta$ where Stokes phenomena occur, cf. Remark 4.5).

5. PROOF OF THE MAIN THEOREMS

In the preceding section we clarified the structure of Stokes phenomena for WKB solutions of Equations (2.1) and (2.21). Making use of this structure of Stokes phenomena and Proposition 3.5, we prove the multisummability of WKB solutions of (2.1) and (2.21) in this section.

5.1. PROOF OF THEOREM 2.1

We first prove the $(4,1)$-multisummability of WKB solutions of (2.1).

Let d be any direction where no Stokes phenomenon occurs with the WKB solutions $\widehat{\psi}_{\pm}^{(0)}(z,\eta)$ of Eq. (2.1). For the sake of definiteness, we here take, for example, $d = 7\pi/16$ and prove the $(4,1)$-multisummability of $\widehat{\psi}_{-}^{(0)}(z,\eta)$ in the direction d.

In this case we have $k_1 = 4$, $k_2 = 1$, $q = 2$ and

$$d \in I_1 = \Big[\frac{5}{16}\pi, \frac{9}{16}\pi\Big] \subset I_2 = \Big[-\frac{1}{16}\pi, \frac{15}{16}\pi\Big]. \tag{5.1}$$

We set $\varphi_0 = d - \pi = -9\pi/16$ and put (i) all the directions where a Stokes phenomenon occurs (we call such directions "singular directions" in what follows), (ii) boundaries of I_μ $(\mu = 1,2)$, and (iii) φ_0, $\varphi_0 + 2\pi$ in the order of magnitude from φ_0 as follows:

$$\phi_0 = \varphi_0 = -\frac{9}{16}\pi,\ \phi_1 = -\frac{1}{2}\pi,\ \phi_2 = -\frac{3}{8}\pi,\ \phi_3 = -\frac{1}{4}\pi,\ \phi_4 = -\frac{1}{16}\pi,$$
$$\phi_5 = 0,\ \phi_6 = \frac{1}{4}\pi,\ \phi_7 = \frac{5}{16}\pi,\ \phi_8 = \frac{1}{2}\pi,\ \phi_9 = \frac{9}{16}\pi,\ \phi_{10} = \frac{5}{8}\pi,$$
$$\phi_{11} = \frac{3}{4}\pi,\ \phi_{12} = \frac{15}{16}\pi,\ \phi_{13} = \pi,\ \phi_{14} = \frac{5}{4}\pi,\ \phi_{15} = \varphi_0 + 2\pi = \frac{23}{16}\pi,$$

In this notation $I_1 = [\phi_7, \phi_9]$ and $I_2 = [\phi_4, \phi_{12}]$. We also express the Stokes phenomena for $\widehat{\psi}_{\pm}^{(0)}$ at the direction ϕ_k in the following way:

$$\psi_*^{(0)} - \widetilde{\psi}_*^{(0)} = \sum_{*'=\pm} c_k(*,*';\alpha)\widetilde{\psi}_{*'}^{(0)} \quad (* = \pm), \tag{5.2}$$

or

$$\psi_*^{(0)} \rightsquigarrow \psi_*^{(0)} + \sum_{*'=\pm} c_k(*,*';\alpha)\psi_{*'}^{(0)}, \tag{5.3}$$

which means $\psi_*^{(0)}$ for $\arg\eta < \phi_k$ is analytically continued to the right-hand side of (5.3) for $\arg\eta > \phi_k$. Here $c_k(*,*';\alpha)$ is a constant (i.e., independent of z) satisfying

$$c_k(*,*';\alpha) = O(\exp(-c|\eta|^\alpha)). \tag{5.4}$$

Note that, if the Stokes phenomena in question are of type A (resp., type B), then $\alpha = 1$ (resp., $\alpha = 4$).

We now define $f(\eta;\theta)$ for $\varphi_0 \le \theta \le \varphi_0 + 2\pi$. First, we define $f(\eta;\theta)$ for $\theta \in [\varphi_0, \varphi_0 + 2\pi] \setminus \{\phi_k\}_{0\le k\le 15}$.

(In $\mathbf{I_1}$) In I_1, i.e., in $[\phi_7, \phi_9]$, taking all the Stokes phenomena into account, we define $f(\eta;\theta)$ as follows:

$$f(\eta;\theta) = \begin{cases} \psi_-^{(0)}(z,\eta;\theta) & \text{for } \phi_7 < \theta < \phi_8, \\ \psi_-^{(0)}(z,\eta;\theta) + c_8(-,-;4)\psi_-^{(0)}(z,\eta;\theta) & \text{for } \phi_8 < \theta < \phi_9, \end{cases} \tag{5.5}$$

where $\psi_{\pm}^{(0)}(z,\eta;\theta)$ designates $\psi_{\pm}^{(0)}(z,\eta)$ defined by the integral along $\Gamma_{\pm}$ when $\arg\eta=\theta$. Note that at $\arg\eta=\phi_8$ we have the following Stokes phenomenon of type B for $\widehat{\psi}_{-}^{(0)}$.

$$\psi_{-}^{(0)} \rightsquigarrow \psi_{-}^{(0)} + c_8(-,-;4)\psi_{-}^{(0)}. \tag{5.6}$$

(In $\mathbf{I_2}\setminus\mathbf{I_1}$) On ∂I_1 we neglect terms of order $O(\exp(-c|\eta|^4))$ and in $I_2\setminus I_1$ we take only Stokes phenomena of type A into account. To be more specific, we define $f(\eta;\theta)$ for $\theta\in[\phi_4,\phi_7]$ by

$$f(\eta;\theta)=\psi_{-}^{(0)}(z,\eta;\theta) \quad \text{for } \phi_4<\theta<\phi_7, \tag{5.7}$$

and for $\theta\in[\phi_9,\phi_{12}]$ by

$$f(\eta;\theta)=\begin{cases}\psi_{-}^{(0)}(z,\eta;\theta) & \text{for } \phi_9<\theta<\phi_{10},\\ \psi_{-}^{(0)}(z,\eta;\theta)+c_{10}(-,+;1)\psi_{+}^{(0)}(z,\eta;\theta) & \text{for } \phi_{10}<\theta<\phi_{12}.\end{cases} \tag{5.8}$$

(In $[\boldsymbol{\varphi_0},\boldsymbol{\varphi_0}+\mathbf{2}\boldsymbol{\pi}]\setminus\mathbf{I_2}$) On ∂I_2 we neglect terms of order $O(\exp(-c|\eta|))$ and in $[\varphi_0,\varphi_0+2\pi]\setminus I_2$ we ignore all the Stokes phenomena, that is, we define $f(\eta;\theta)$ simply by

$$f(\eta;\theta)=\psi_{-}^{(0)}(z,\eta;\theta) \quad \text{for } \phi_0<\theta<\phi_4 \text{ and } \phi_{12}<\theta<\phi_{15}. \tag{5.9}$$

Finally, we define $f(\eta;\theta)$ for $\theta=\phi_k$ by the following:

$$f(\eta;\phi_k)=\begin{cases} f(\eta;\phi_k+\delta) & \text{when } \varphi_0\le\phi_k<d,\\ f(\eta;\phi_k-\delta) & \text{when } d<\phi_k\le\varphi_0+2\pi,\end{cases} \tag{5.10}$$

where δ is a sufficiently small positive number.

Note that, as the number of singular directions is finite, by taking sufficiently small ϵ and ρ we may assume that $f(\eta;\theta)$ is analytic in a sector $S(\theta,\epsilon,\rho)$. Further, we may also assume that every $S(\theta,\epsilon,\rho)$ contains at most one ϕ_k. Hence the conditions (i) and (ii) in Proposition 3.5 are satisfied. By the above definition of $f(\eta;\theta)$ conditions (iv) and (v) also hold. Thus it suffices to check condition (iii).

(I) (*When* $\theta_1,\theta_2\in I_1$.) Since all the Stokes phenomena are taken into account in I_1, $f(\eta;\theta_1)=f(\eta;\theta_2)$ trivially holds.

(II) (*When* $\theta_1,\theta_2\in I_2$ *and one of* θ_1 *and* θ_2 *does not belong to* I_1.) Discontinuity for $f(\eta;\theta)$ is observed only at $\arg\eta=\phi_5,\phi_6,\phi_9$ and ϕ_{11}.

First, since Stokes phenomena at $\arg\eta=\phi_5,\phi_6$ are of type B, condition (3.10)′ is satisfied near $\arg\eta=\phi_5,\phi_6$.

Next, as ϕ_9 is a boundary point of I_1 and no Stokes phenomenon occurs there, at $\arg\eta=\phi_9$ we have

$$\begin{aligned} &f(\eta;\phi_9+\delta)-f(\eta;\phi_9-\delta) \qquad (5.11)\\ &\quad=\psi_{-}^{(0)}(z,\eta;\phi_9+\delta)-(\psi_{-}^{(0)}(z,\eta;\phi_9-\delta)+c_8(-,-;4)\psi_{-}^{(0)}(z,\eta;\phi_9-\delta))\\ &\quad=-c_8(-,-;4)\psi_{-}^{(0)}(z,\eta;\phi_9-\delta)=O(\exp(-c|\eta|^4))\end{aligned}$$

for a small positive constant δ. Note that, since the term $c_8\psi_-^{(0)}$ appears in the Stokes phenomenon at $\arg\eta = \phi_8$, it precisely satisfies

$$c_8\psi_-^{(0)} = O(\exp(-c(\eta e^{-i\phi_8})^4)). \tag{5.12}$$

Hence (5.11) is valid at $\arg\eta = \phi_9$ in view of $|\phi_9 - \phi_8| < \pi/8 = \pi/(2k_1)$.

Finally, at ϕ_{11} we have

$$\begin{aligned}
&f(\eta;\phi_{11}+\delta) - f(\eta;\phi_{11}-\delta) \\
&\quad = (\psi_-^{(0)}(z,\eta;\phi_{11}+\delta) + c_{10}(-,+;1)\psi_+^{(0)}(z,\eta;\phi_{11}+\delta)) \\
&\qquad - (\psi_-^{(0)}(z,\eta;\phi_{11}-\delta) + c_{10}(-,+;1)\psi_+^{(0)}(z,\eta;\phi_{11}-\delta)) \\
&\quad = \psi_-^{(0)}(z,\eta;\phi_{11}-\delta) - \psi_-^{(0)}(z,\eta;\phi_{11}+\delta) = O(\exp(-c|\eta|^4)),
\end{aligned} \tag{5.13}$$

since a Stokes phenomenon for $\psi_-^{(0)}$ at $\arg\eta = \phi_{11}$ is of type B and no Stokes phenomenon occurs with $\psi_+^{(0)}$ there.

(III) (*When* $\theta_1, \theta_2 \in [\varphi_0, \varphi_0 + 2\pi]$ *and one of* θ_1 *and* θ_2 *does not belong to* I_2.) Discontinuity for $f(\eta;\theta)$ is observed only at $\arg\eta = \phi_{12}, \phi_{13}$ and ϕ_{14}.

At ϕ_{12}, by an argument similar to the one at ϕ_9, we have

$$\begin{aligned}
&f(\eta;\phi_{12}+\delta) - f(\eta;\phi_{12}-\delta) \\
&\quad = \psi_-^{(0)}(z,\eta;\phi_{12}+\delta) - (\psi_-^{(0)}(z,\eta;\phi_{12}-\delta) + c_{10}(-,+;1)\psi_+^{(0)}(z,\eta;\phi_{12}-\delta)) \\
&\quad = -c_{10}(-,+;1)\psi_+^{(0)}(z,\eta;\phi_{12}-\delta) = O(\exp(-c|\eta|)).
\end{aligned} \tag{5.14}$$

Note again that, since the term $c_{10}\psi_+^{(0)}$ appears in the Stokes phenomenon at $\arg\eta = \phi_{10}$, it satisfies

$$c_{10}\psi_+^{(0)} = O(\exp(-c(\eta e^{-i\phi_{10}}))) \tag{5.15}$$

and hence (5.14) is valid at $\arg\eta = \phi_{12}$ in view of $|\phi_{12} - \phi_{10}| < \pi/2 = \pi/(2k_2)$.

Since Stokes phenomena at $\arg\eta = \phi_{13}, \phi_{14}$ are of type B, the required condition is satisfied also near $\arg\eta = \phi_{13}, \phi_{14}$.

We have thus checked all conditions in Proposition 3.5. Therefore the $(4,1)$-multisummability of the WKB solution $\widehat{\psi}_-^{(0)}$ of Equation (2.1) is now verified thanks to Proposition 3.5 (cf. Remark 3.6).

5.2. PROOF OF THEOREM 2.2

Theorem 2.2 is proved by a similar argument as in the preceding subsection. We explain an outline of the proof of Theorem 2.2 in this subsection.

Let d be any direction where no Stokes phenomenon occurs with the WKB solutions $\widehat{\psi}_j^{(0)}(z,\eta)$ $(j = 1,2,3)$ of Eq. (2.21). Here we take $d = 7\pi/32$ and prove the $(8,5,1)$-multisummability of $\widehat{\psi}_1^{(0)}$ in the direction d.

In this case we have $k_1 = 8$, $k_2 = 5$, $k_3 = 1$, $q = 3$ and

$$d \in I_1 = \Big[\frac{5}{32}\pi, \frac{9}{32}\pi\Big] \subset I_2 = \Big[\frac{19}{160}\pi, \frac{51}{160}\pi\Big] \subset I_3 = \Big[-\frac{9}{32}\pi, \frac{23}{32}\pi\Big]. \tag{5.16}$$

We set $\varphi_0 = d - \pi = -25\pi/32$ and again put all the singular directions, boundaries of I_μ $(\mu = 1, 2, 3)$, φ_0 and $\varphi_0 + 2\pi$ in the order of magnitude from φ_0 so that

$$\begin{aligned} &\phi_0 = \varphi_0, \quad \phi_{19} < d < \phi_{20}, \quad \phi_{39} = \varphi_0 + 2\pi, \\ &I_1 = [\phi_{17}, \phi_{20}], \quad I_2 = [\phi_{16}, \phi_{23}], \quad I_3 = [\phi_{10}, \phi_{31}]. \end{aligned} \tag{5.17}$$

Similarly to (5.3), we express the Stokes phenomena for $\widehat{\psi}_j^{(0)}$ $(j = 1, 2, 3)$ at the direction ϕ_k as

$$\psi_j^{(0)} \rightsquigarrow \psi_j^{(0)} + \sum_{1 \le j' \le 3} c_k(j, j'; \alpha) \psi_{j'}^{(0)}, \tag{5.18}$$

where $c_k(j, j'; \alpha)$ is a constant satisfying

$$c_k(j, j'; \alpha) = O(\exp(-c|\eta|^\alpha)). \tag{5.19}$$

If Stokes phenomena in question are of type A (resp., type B, type C), then $\alpha = 1$ (resp., $\alpha = 5$, $\alpha = 8$).

We then define $f(\eta; \theta)$ for $\varphi_0 \le \theta \le \varphi_0 + 2\pi$ as follows:

(In $\mathbf{I_1}$) In I_1, taking all the Stokes phenomena into account, we define

$$\begin{aligned} f(\eta; \theta) = {} & \psi_1^{(0)}(z, \eta; \theta) + c_{18}(1, 2; 8)\psi_2^{(0)}(z, \eta; \theta) \\ & + c_{19}(1, 1; 5)\big(\psi_1^{(0)}(z, \eta; \theta) + c_{18}(1, 2; 8)\psi_2^{(0)}(z, \eta; \theta)\big) \\ & + c_{19}(1, 2; 5)\big(\psi_2^{(0)}(z, \eta; \theta) + c_{18}(2, 2; 8)\psi_2^{(0)}(z, \eta; \theta)\big) \end{aligned} \tag{5.20}$$

for $\phi_{17} < \theta < \phi_{18}$,

$$f(\eta; \theta) = \psi_1^{(0)}(z, \eta; \theta) + c_{19}(1, 1; 5)\psi_1^{(0)}(z, \eta; \theta) + c_{19}(1, 2; 5)\psi_2^{(0)}(z, \eta; \theta) \tag{5.21}$$

for $\phi_{18} < \theta < \phi_{19}$, and

$$f(\eta; \theta) = \psi_1^{(0)}(z, \eta; \theta) \tag{5.22}$$

for $\phi_{19} < \theta < \phi_{20}$.

(In $\mathbf{I_2 \setminus I_1}$) On ∂I_1 we neglect terms of order $O(\exp(-c|\eta|^8))$ and in $I_2 \setminus I_1$, ignoring Stokes phenomena of type C, we take account of Stokes phenomena of type A and type B only. That is, we define $f(\eta; \theta)$ for $\theta \in [\phi_{16}, \phi_{17}]$ by

$$f(\eta; \theta) = \psi_1^{(0)}(z, \eta; \theta) + c_{19}(1, 1; 5)\psi_1^{(0)}(z, \eta; \theta) + c_{19}(1, 2; 5)\psi_2^{(0)}(z, \eta; \theta), \tag{5.23}$$

and for $\theta \in [\phi_{20}, \phi_{23}]$ by

$$f(\eta; \theta) = \begin{cases} \psi_1^{(0)}(z, \eta; \theta) & \text{for } \phi_{20} < \theta < \phi_{21}, \\ \psi_1^{(0)}(z, \eta; \theta) + c_{21}(1, 3; 1)\psi_3^{(0)}(z, \eta; \theta) & \text{for } \phi_{21} < \theta < \phi_{23}. \end{cases} \tag{5.24}$$

(In $\mathbf{I_3 \setminus I_2}$) On ∂I_2 we neglect terms of order up to $O(\exp(-c|\eta|^5))$ and in $I_3 \setminus I_2$ we take only Stokes phenomena of type A into account. Consequently, we define $f(\eta;\theta)$ by

$$f(\eta;\theta) = \begin{cases} \psi_1^{(0)}(z,\eta;\theta) & \text{for } \phi_{10} < \theta < \phi_{16}, \\ \psi_1^{(0)}(z,\eta;\theta) + c_{21}(1,3;1)\psi_3^{(0)}(z,\eta;\theta) & \text{for } \phi_{23} < \theta < \phi_{31}. \end{cases} \tag{5.25}$$

(In $[\varphi_0, \varphi_0 + 2\pi] \setminus \mathbf{I_3}$) On ∂I_3 we neglect terms of order up to $O(\exp(-c|\eta|))$ and in $[\varphi_0, \varphi_0 + 2\pi] \setminus I_2$ we ignore all the Stokes phenomena, that is, we define $f(\eta;\theta)$ simply by

$$f(\eta;\theta) = \psi_-^{(0)}(z,\eta;\theta) \quad \text{for } \phi_0 < \theta < \phi_{10} \text{ and } \phi_{31} < \theta < \phi_{39}. \tag{5.26}$$

Finally, we define $f(\eta;\theta)$ for $\theta = \phi_k$ by the following:

$$f(\eta;\phi_k) = \begin{cases} f(\eta;\phi_k + \delta) & \text{when } \varphi_0 \le \phi_k < d, \\ f(\eta;\phi_k - \delta) & \text{when } d < \phi_k \le \varphi_0 + 2\pi. \end{cases} \tag{5.27}$$

Defining $f(\eta;\theta)$ as above, we can confirm conditions (i), (ii), (iv) and (v) in Proposition 3.5 by taking sufficiently small ϵ and ρ. Furthermore, we can check condition (iii) also by similar arguments as in the preceding subsection. As the arguments are completely similar, we omit them here.

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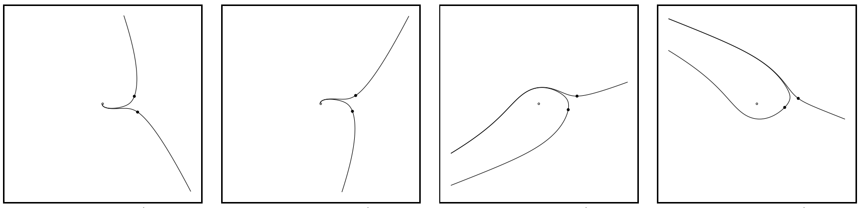
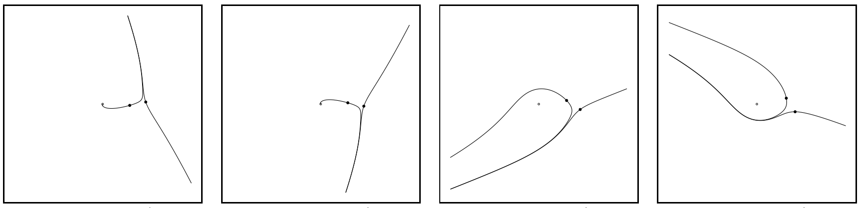
APPENDIX

A. FIGURES OF STEEPEST DESCENT PATHS FOR EQ. (2.1)

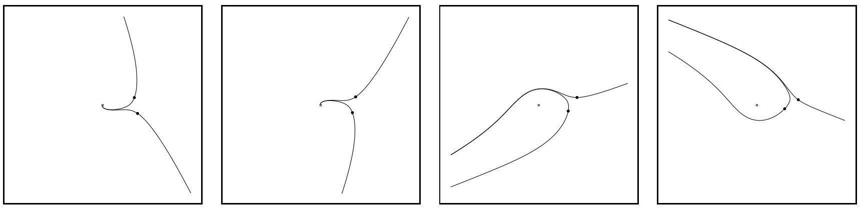
In Appendix A we present figures of steepest descent paths for Eq. (2.1), that is, those of (2.9), for several different values of $\arg\eta \in [0, 2\pi)$. (Here we take $|\eta| = 10$.)



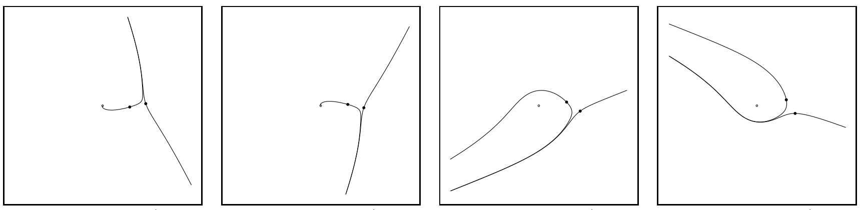
$\arg\eta = 6\pi/100$ $\arg\eta = 19\pi/100$ $\arg\eta = 31\pi/100$ $\arg\eta = 44\pi/100$



$\arg\eta = 56\pi/100$ $\arg\eta = 69\pi/100$ $\arg\eta = 81\pi/100$ $\arg\eta = 94\pi/100$



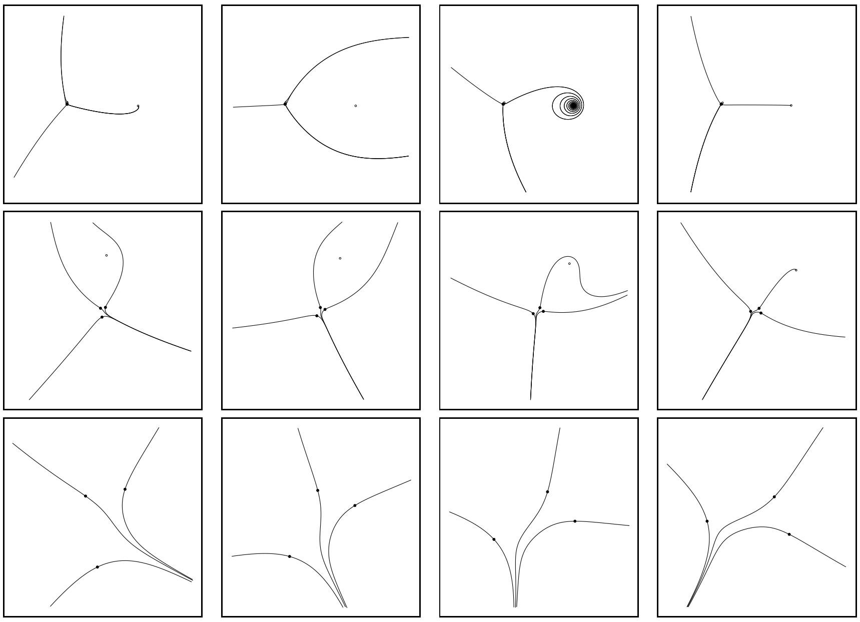
$\arg\eta = 106\pi/100$ $\arg\eta = 119\pi/100$ $\arg\eta = 131\pi/100$ $\arg\eta = 144\pi/100$



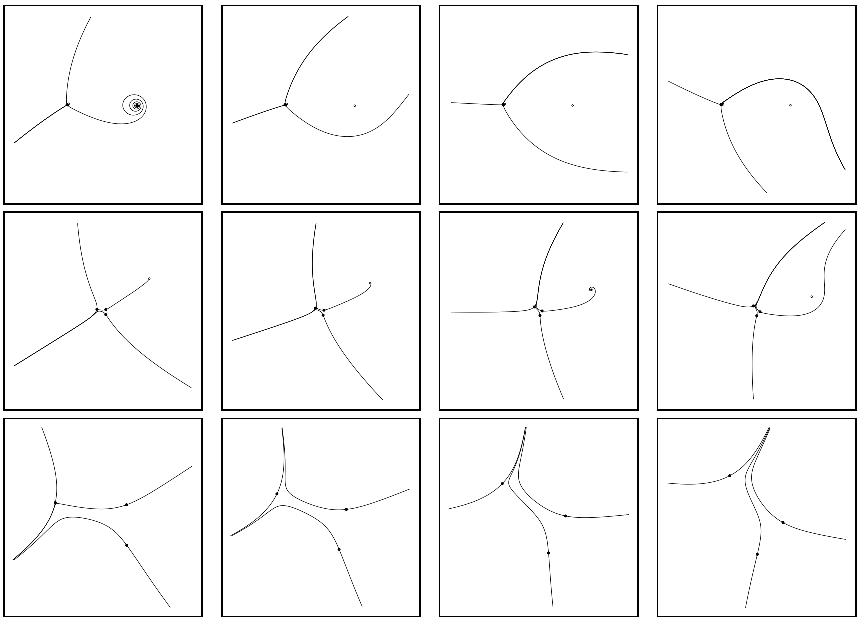
$\arg\eta = 156\pi/100$ $\arg\eta = 169\pi/100$ $\arg\eta = 181\pi/100$ $\arg\eta = 194\pi/100$

B. FIGURES OF STEEPEST DESCENT PATHS FOR EQ. (2.21)

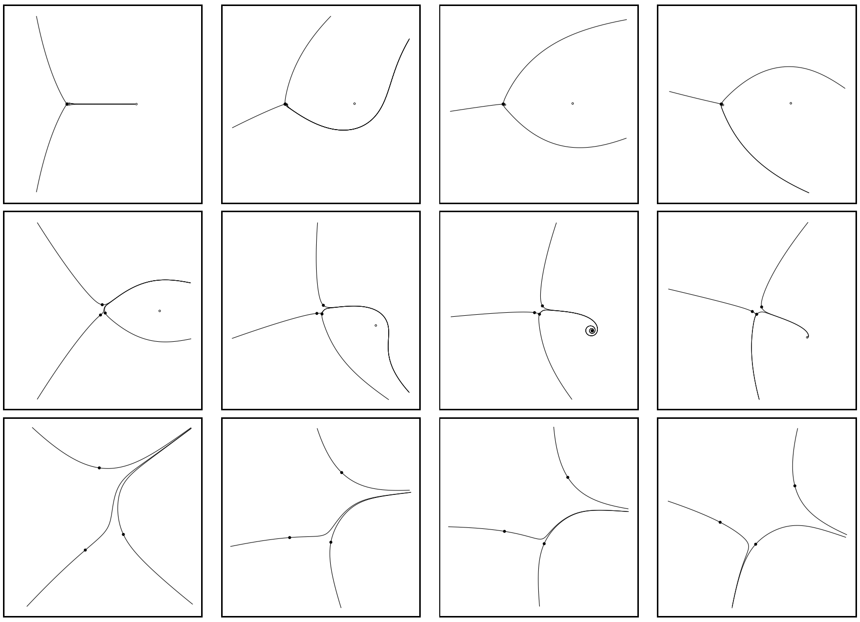
In Appendix B we show figures of steepest descent paths for Eq. (2.21), that is, those of (2.25) or equivalently (2.27), for several different values of $\arg\eta \in [0, 2\pi)$. (Here we take $|\eta| = 20$.) To distinguish the saddle points t_j $(j = 1, 2, 3)$ and t_1^{sing}, we also present magnified ones: Figures at the top are original ones of (2.27), those in the middle are magnified ones and those at the bottom are more magnified ones.



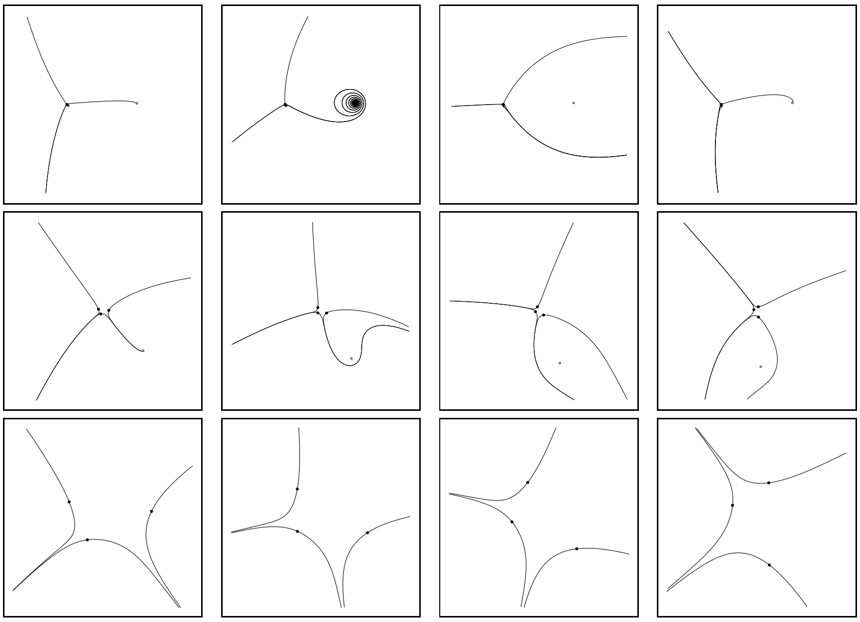
$\arg\eta = 3\pi/100$ $\arg\eta = 12\pi/100$ $\arg\eta = 19\pi/100$ $\arg\eta = 25\pi/100$



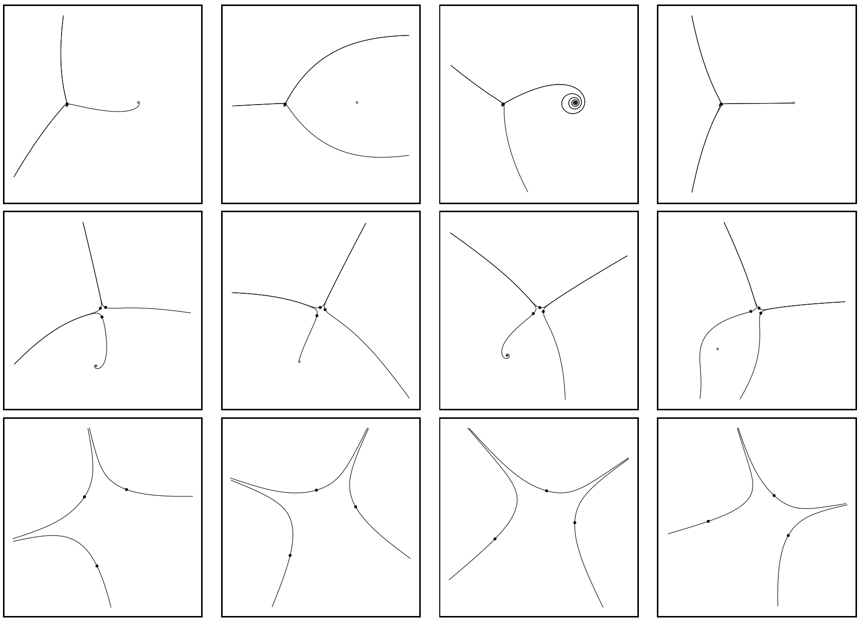
$\arg\eta = 31\pi/100$ $\arg\eta = 34\pi/100$ $\arg\eta = 38\pi/100$ $\arg\eta = 42\pi/100$



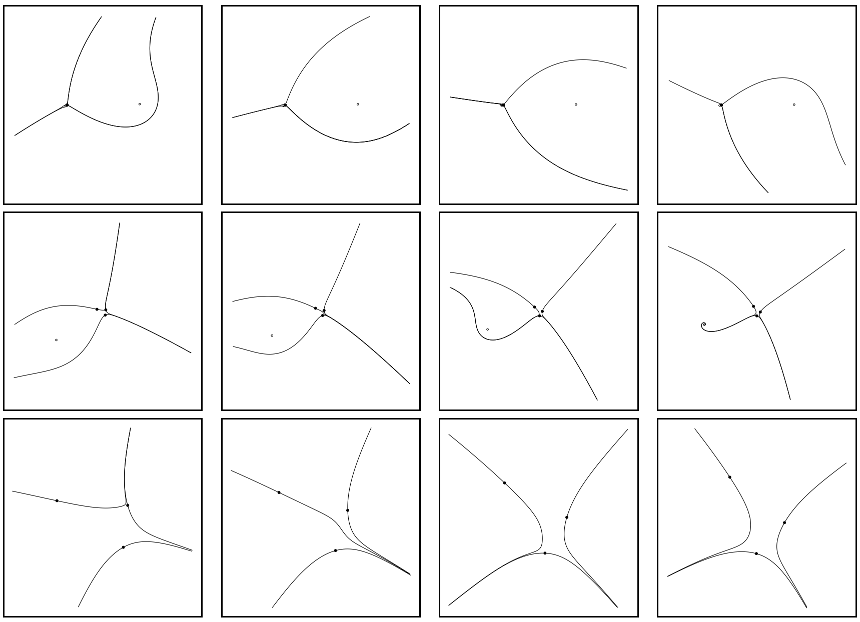
$\arg\eta = 50\pi/100$ $\arg\eta = 58\pi/100$ $\arg\eta = 61\pi/100$ $\arg\eta = 65\pi/100$



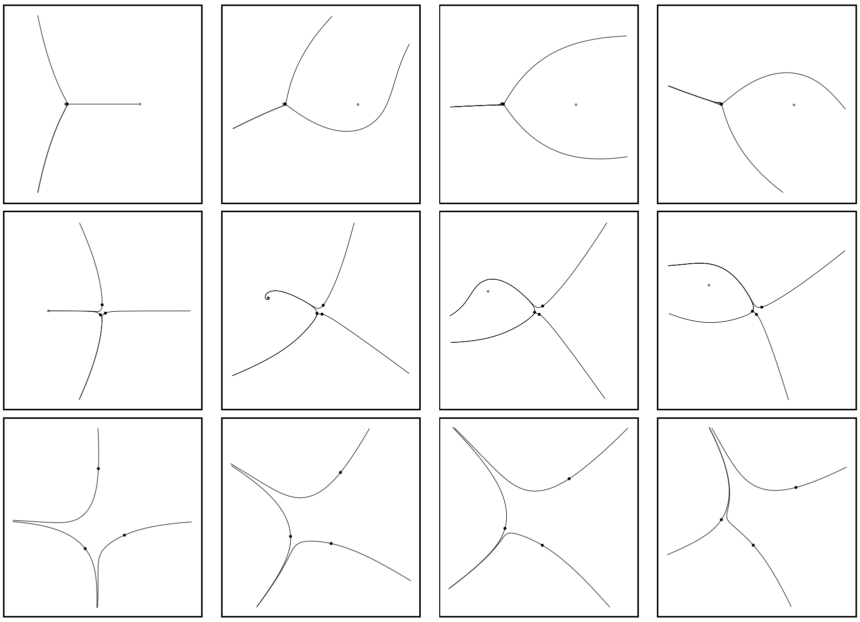
$\arg\eta = 74\pi/100$ $\arg\eta = 81\pi/100$ $\arg\eta = 87\pi/100$ $\arg\eta = 97\pi/100$



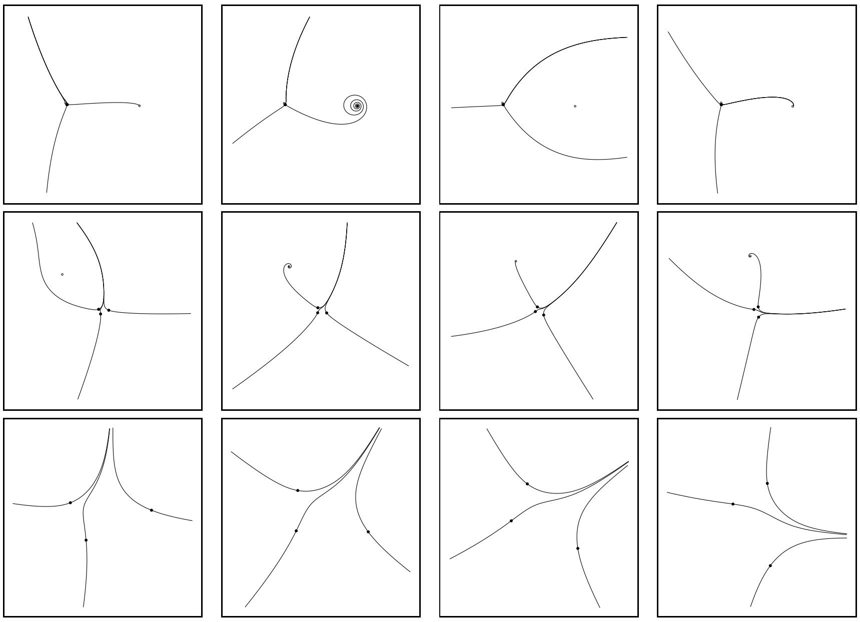
$\arg \eta = 103\pi/100$ $\arg \eta = 112\pi/100$ $\arg \eta = 119\pi/100$ $\arg \eta = 125\pi/100$



$\arg \eta = 132\pi/100$ $\arg \eta = 135\pi/100$ $\arg \eta = 139\pi/100$ $\arg \eta = 142\pi/100$



$\arg \eta = 150\pi/100$ $\arg \eta = 158\pi/100$ $\arg \eta = 162\pi/100$ $\arg \eta = 166\pi/100$



$\arg \eta = 174\pi/100$ $\arg \eta = 181\pi/100$ $\arg \eta = 187\pi/100$ $\arg \eta = 197\pi/100$

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